

Fix-finite approximation property in F -spaces

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Abstract

In this paper, with the aid of the simplicial approximation property, the Hopf's construction and Dugundji's homotopy extension Theorem, we first show that if C is a nonempty compact convex subset of an F -space $(E, \|\cdot\|)$, then for every $\varepsilon > 0$ and every subset D of E containing C and every continuous map $f : D \rightarrow C$ there exists a continuous map $g : D \rightarrow C$ which is ε -near to f and has only a finite number of fixed points. Secondly, by using this result and the simplicial approximation property, we establish that for any $\varepsilon > 0$ and every path and simply connected compact subset D of E containing C and for each continuous n -valued multifunction $F : D \rightarrow 2^C$ there exists a continuous n -valued multifunction $G : D \rightarrow 2^C$ which is ε -near to F and has only a finite number of fixed points.

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1 Introduction

Let (X, d) be a metric space, A and D two nonempty subsets of X such that $A \subset D$. We say that the pair (D, A) enjoy the fix-finite approximation property (F.F.A.P) for a family \mathcal{F} of maps $f : D \rightarrow A$ or multifunctions $F : D \rightarrow 2^A$ (where 2^A is the set of all nonempty subsets of A) if for every $\varepsilon > 0$ there exists $g \in \mathcal{F}$ which is ε -near to f and has only a finite number of fixed points (respectively there exists a multifunction $G : D \rightarrow 2^A$ in \mathcal{F} which is ε -near to F and has only a finite number of fixed points). During the last century several authors studies the fix-finite approximation property: Hopf [8], Baillon and Rallis [1] and Schirmer [11]. Later on, in [11, 12] we established some results concerning the fix-finite approximation property in the cases of normed vector spaces and metrizable locally

convex vector spaces. In these cases we have used as key result the Schauder mapping (for examples see: [5, 12, 13]). In the present paper, we cannot use the Schauder mapping because in F -spaces we haven't the useful property of partition of the unity. However, in this paper we can use the simplicial approximation property which was proved by Dobrowolski in [6] for compact convex sets in F -spaces.

In the present paper, we consider the more general case of F -spaces. Our key result in this paper is the following: if C is a nonempty compact convex subset of an F -space $(E, || \quad ||)$, then for every $\varepsilon > 0$ and every subset D of E containing C and every continuous map $f : D \rightarrow C$ there exists a continuous map $g : D \rightarrow C$ which is ε -near to f and has only a finite number of fixed points (Theorem 3.1). By using this result and the simplicial approximation property, we establish: if C is a nonempty compact convex subset of an F -space $(E, || \quad ||)$, then for every $\varepsilon > 0$ and for every and every path and simply connected compact subset D of E containing C and for every continuous n -valued multifunction $F : D \rightarrow 2^C$ there exists a continuous n -valued multifunction $G : D \rightarrow 2^C$ which is ε -near to F and has only a finite number of fixed points (Theorem 4.1).

2 Preliminaries

In this section, we shall recall some definitions and well-know results for subsequent use.

Let $\varepsilon > 0$ and let X be a topological space and (Y, d) be a metric space. We say that $f : X \rightarrow Y$ is a map if it is a single valued function.

Two continuous maps f and g from X to Y are said to be ε -near if

$$d(f(x), g(x)) < \varepsilon, \text{ for all } x \in X.$$

A homotopy $h_t : X \rightarrow Y, (0 \leq t \leq 1)$ is said to be an ε -homotopy if

$$\sup\{d(h_t(x), h_{t'}(x)) : t, t' \in [0, 1]\} < \varepsilon, \text{ for all } x \in X.$$

Two continuous maps f and g from X to Y are said to be ε -homotopic if there exists an ε -homotopy $(h_t)_{t \in [0, 1]}$ from X to Y such that $h_0 = f$ and $h_1 = g$.

Let X be a Hausdorff topological space and $f : X \rightarrow X$ be a map. A point x of X is said to be a fixed point of f if $f(x) = x$. We denote by $Fix(f)$ the set of all fixed points of f .

Let Y be a metric space. One says that Y is an absolute neighborhood retract (ANR) if for any nonempty closed subset A of an arbitrary metric space X and for any continuous map $f : A \rightarrow Y$, then there exists an open subset U of X containing A and a continuous map $g : U \rightarrow Y$ which is an extension of f (i.e. $g(x) = f(x)$, for all $x \in A$).

In [7], Dugundji established the homotopy extension Theorem for ANR's.

Theorem 2.1 *Let X be a metrizable space and Y a ANR. For $\varepsilon > 0$, there exists $\delta > 0$ such that for any two δ -near maps $f, g : X \rightarrow Y$ and δ -homotopy $j_t : A \rightarrow Y$, where A is a closed subspace of X and $j_0 = f|_A$, $j_1 = g|_A$, there exists an ε -homotopy $h_t : X \rightarrow Y$ such that $h_0 = f$, $h_1 = g$ and $h_{t|_A} = j_t$ for all $t \in [0, 1]$.*

Let X and Y be two Hausdorff topological spaces. A multifunction $F : X \rightarrow Y$ is a map from X into the set 2^Y of nonempty subsets of Y . The range of F is $F(X) = \bigcup_{x \in X} F(x)$.

The multifunction $F : X \rightarrow Y$ is said to be upper semi-continuous (usc) if for each open subset V of Y with $F(x) \subset V$ there exists an open subset U of X with $x \in U$ and $F(U) \subset V$.

The multifunction $F : X \rightarrow Y$ is called lower semi-continuous (lsc) if for every $x \in X$ and open subset V of Y with $F(x) \cap V \neq \emptyset$ there exists an open subset U of X with $x \in U$ and $F(x') \cap V \neq \emptyset$ for all $x' \in U$.

The multifunction $F : X \rightarrow Y$ is continuous if it is both upper semi-continuous and lower semi-continuous. The multifunction F is compact if it is continuous and the closure of its range $\overline{F(X)}$ is a compact subset of Y .

A point x of X is said to be a fixed point of a multifunction $F : X \rightarrow X$ if $x \in F(x)$. We denote by $Fix(F)$ the set of all fixed points of F .

Let (X, d) be a metric space. We denote by $C(X)$ the set of nonempty compact subsets of X . Let A and B be two elements of $C(X)$. The Hausdorff distance between A and B , $d_H(A, B)$, is defined by setting:

$$d_H(A, B) = \max \{ \rho(A, B), \rho(B, A) \}$$

where

$$\rho(A, B) = \sup \{ d(x, B) : x \in A \},$$

$$\rho(B, A) = \sup \{ d(y, A) : y \in B \}$$

and

$$d(x, B) = \inf \{ d(x, y) : y \in B \}.$$

Let (X, τ) be a Hausdorff topological space and (Y, d) be a metric space. Let F and G be two compact multifunctions from X to Y . We define the Hausdorff distance between F and G by setting:

$$d_H(F, G) = \sup \{ d_H(F(x), G(x)) : x \in X \}.$$

Let $\varepsilon > 0$ and F and G be two compact multifunctions from X to Y . We say that F and G are ε -near if $d_H(F, G) < \varepsilon$.

A multifunction $F : X \rightarrow X$ is said to be an n -function if there exist n continuous maps $f_i : X \rightarrow X$, where $i = 1, \dots, n$, such that $F(x) = \{f_1(x), \dots, f_n(x)\}$ and $f_i(x) \neq f_j(x)$ for all $x \in X$ and $i, j = 1, \dots, n$ with $i \neq j$.

Definition 2.2 *Let X and Y be two Hausdorff topological spaces. A multifunction $F : X \rightarrow Y$ is said to be n -valued if for all $x \in X$, the subset $F(x)$ of Y consists of n points.*

Let X and Y be two Hausdorff topological spaces and let $F : X \rightarrow Y$ be a n -valued continuous multifunction. Then, we can write $F(x) = \{y_1, \dots, y_n\}$ for all $x \in X$. We define a real function γ on X by

$$\gamma(x) = \min \{ \|y_i - y_j\| : y_i, y_j \in F(x), i, j = 1, \dots, n, i \neq j \}, \text{ for all } x \in X.$$

Then, the gap of F is defined by $\gamma(F) = \inf \{ \gamma(x) : x \in X \}$. From [13, p.76], the function γ is continuous. So if X is compact, then $\gamma(F) > 0$.

In this paper, we shall need the following result due to H. Schirmer [11].

Lemma 2.3 [11]. *Let X and Y be two compact Hausdorff topological spaces. If X is path and simply connected and $F : X \rightarrow Y$ is a continuous n -valued multifunction, then F is an n -function.*

Next we recall the definition of an F -norm.

Definition 2.4 [10]. *Let E be a real or a complex linear vector space. A non-negative function $\|\cdot\| : E \rightarrow \mathbb{R}$ is said to be an F -norm if it satisfies the following properties:*

- (1) $\forall x, y \in E, \|x + y\| \leq \|x\| + \|y\|$;
- (2) $\|x\| = 0 \Leftrightarrow x = o_E$;
- (3) $\forall x \in E$ and $\forall t \in [-1, 1], \|tx\| \leq \|x\|$;
- (4) if $\alpha_n \rightarrow 0$, then for every $x \in E$ $\alpha_n x \rightarrow o_E$;
- (5) the metric $d(x, y) = \|x - y\|$ is complete.

Throughout this paper we assume that E is a topological vector space that is not necessarily locally convex. A linear vector space equipped with an F -norm is called an F -space.

In [10] Kalton and al introduced the simplicial approximation property which is a useful tool for our results in this paper.

Definition 2.5 [10]. *A convex subset C of a metric linear space $(E, \|\cdot\|)$ has the simplicial approximation property if, for every $\varepsilon > 0$, there exists a finite-dimensional compact convex $C_\varepsilon \subset C$ such that, if S is any finite-dimensional simplex in C , then there exists a continuous map $h : S \rightarrow C_\varepsilon$ with $\|h(x) - x\| < \varepsilon$ for every $x \in S$.*

Recently in [6, Lemma 2.2] and [6, Corollary 2.6] Dobrowolski proved the following interesting result.

Lemma 2.6 [6]. *Every compact convex set in a metric linear space has the simplicial approximation property.*

An equivalent formulation of the simplicial approximation property is given in [10, Theorem 9.8].

Lemma 2.7 [10]. *If K is an infinite-dimensional compact convex set in an F -space $E = (E, \|\cdot\|)$, then the following statements are equivalent:*

- (1) *K has the simplicial approximation property,*
- (2) *if $\varepsilon > 0$, there exist a simplex S_ε in K and a continuous map $h_\varepsilon : K \rightarrow S_\varepsilon$ such that $\|h_\varepsilon(x) - x\| < \varepsilon$ for every $x \in K$.*

3 The key result

In this section with the aid of the simplicial approximation property (Lemma 2.7), the Hopf's construction [3, 8] and Dugundji's homotopy extension Theorem [7] we shall give our key result in this paper.

Theorem 3.1 *Let C be a nonempty compact convex subset of an F -space $(E, \|\cdot\|)$, D a subset of E containing C , $f : D \rightarrow C$ be a continuous map and $\varepsilon > 0$ be given. Then there exists a continuous map $g : D \rightarrow C$ which is ε -near to f and has only a finite number of fixed points.*

Proof. Let C be a nonempty compact convex subset of an F -space $(E, \|\cdot\|)$, D a subset of E containing C , $f : D \rightarrow C$ be a continuous map and $\varepsilon > 0$ be given. From Lemma 2.7, there exist a simplex S_ε in C and a continuous map $h_\varepsilon : C \rightarrow S_\varepsilon$ such that $\|h_\varepsilon(x) - x\| < \frac{\varepsilon}{2}$ for every $x \in C$. Put $f_\varepsilon = h_\varepsilon \circ f : D \rightarrow S_\varepsilon$. Hence, f_ε is $\frac{\varepsilon}{2}$ -near to f . As S_ε is a simplex, then it is a compact ANR [2]. From Theorem 2.1, for $\frac{\varepsilon}{2} > 0$, there exists $\delta > 0$ such that for any two continuous maps $v, u : S_\varepsilon \rightarrow S_\varepsilon$ δ -near and a δ -homotopy $j_t : S_\varepsilon \rightarrow S_\varepsilon$ with $j_0 = v|_{S_\varepsilon}$ and $j_1 = u|_{S_\varepsilon}$, there exists an $\frac{\varepsilon}{2}$ -homotopy $g_t : D \rightarrow S_\varepsilon$ such that $g_0 = v$ and $g_1 = u$ and $g_{t|_{S_\varepsilon}} = j_t$ for all $t \in [0, 1]$. By [3, p.40] for $\frac{\delta}{2} > 0$, there exists $\lambda > 0$ such that if $\varphi, \psi : S_\varepsilon \rightarrow S_\varepsilon$ are two continuous maps λ -near, then φ and ψ are $\frac{\delta}{2}$ -homotopic.

Since $f_\varepsilon|_{S_\varepsilon}$ is a continuous map and S_ε is a finite polyhedron, then by Hopf's construction [3, 8] there exists a continuous map $k : S_\varepsilon \rightarrow S_\varepsilon$ which is λ -near to $f_\varepsilon|_{S_\varepsilon}$ and has only a finite number of fixed points. Hence, from [3, p.40], k and $f_\varepsilon|_{S_\varepsilon}$ are $\frac{1}{2}\delta$ -homotopic. Let $(h_t)_{(t \in [0, 1])}$ be this $\frac{1}{2}\delta$ -homotopy between k and $f_\varepsilon|_{S_\varepsilon}$ such that $h_0 = f_\varepsilon|_{S_\varepsilon}$ and $h_1 = k$. Now, we define a new homotopy $(j_t)_{t \in [0, 1]}$ by setting:

$$j_t = \begin{cases} h_{2t} & \text{if } 0 \leq t < \frac{1}{2} \\ h_{2t-2} & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

So, we deduce that $(j_t)_{t \in [0,1]}$ is a δ -homotopy such that $j_0 = j_1 = f_\varepsilon|_{S_\varepsilon}$ and $j_{\frac{1}{2}} = k$.

Hence, by Theorem 2.1 there exists an $\frac{1}{2}\varepsilon$ -homotopy $g_t : D \rightarrow S_\varepsilon$ such that $g_0 = g_1 = f$ and $g_t|_{S_\varepsilon} = j_t$ for all $t \in [0, 1]$. Put, $g_{\frac{1}{2}} = g$. Then, $g : D \rightarrow S_\varepsilon$ is a continuous map and $\text{Fix}(g) = \text{Fix}(k)$. Then, $\text{Fix}(g)$ is a finite set. Since for every $x \in C$ we have

$$\|f(x) - g(x)\| \leq \|f(x) - f_\varepsilon(x)\| + \|f_\varepsilon(x) - g(x)\|.$$

So, we get $\|f(x) - g(x)\| < \varepsilon$. Thus, the map g is ε -near to f and has only a finite number of fixed points.

As a consequence of Theorem 3.1, we get the following result.

Corollary 3.2 *Let C be a nonempty compact convex subset of an F -space $(E, \|\cdot\|)$, $f : C \rightarrow C$ be a continuous map and $\varepsilon > 0$ be given. Then there exists a continuous map $f_\varepsilon : C \rightarrow C$ which is ε -near to f and has only a finite number of fixed points.*

4 Fix–finite approximation propriety for n –valued multifunctions

In this section, By using the simplicial approximation property (Lemma 2.7) and Theorem 3.1, we shall establish a fix–finite approximation result for n –valued multifunctions in F –spaces. More precisely we shall show the following.

Theorem 4.1 *Let C be a nonempty compact convex subset of an F -space $(E, \|\cdot\|)$ and let D be a path and simply connected compact subset of E containing C . Then the pair (D, C) satisfies the F.F.A.P. for every continuous n –valued continuous multifunction $F : D \rightarrow 2^C$.*

In order to give the prove of Theorem 4.1, we shall need the following proposition.

Proposition 4.2 *Let C be a nonempty compact convex subset of an F -space $(E, \|\cdot\|)$ and let D be a compact subset of E containing C . Then the pair (D, C) satisfies the F.F.A.P. for any n –function $G : D \rightarrow 2^C$.*

Proof. Let $G : D \rightarrow 2^C$ be an n -function. Then, there exist n continuous maps $g_i : D \rightarrow C$ such that $G(x) = \{g_1(x), \dots, g_n(x)\}$ for all $x \in D$ and $g_i(x) \neq g_j(x)$ for all $x \in D$ and $i, j = 1, \dots, n$ with $i \neq j$.

For all $i, j = 1, \dots, n$ with $i \neq j$, we define $\gamma_{(i,j)}(G) = \min\{\|g_i(x) - g_j(x)\| : x \in D\}$. As every g_i is continuous for all $i = 1, \dots, n$ and D is compact, then for every $i, j = 1, \dots, n$ with $i \neq j$, we have $\gamma_{(i,j)}(G) > 0$. Therefore,

$$\gamma(G) = \min\{\gamma_{(i,j)}(G) : i, j = 1, \dots, n, i \neq j\} > 0.$$

Let $\varepsilon > 0$ be given. Then, we set $\gamma = \min(\frac{1}{2}\gamma(G), \frac{1}{2}\varepsilon)$. Hence, $\gamma > 0$. By Theorem 3.1, for each $i = 1, \dots, n$, there exists a continuous map $h_i : D \rightarrow C$ which is γ -near to g_i and has only a finite number of fixed points. Let $H : D \rightarrow C$ be the multifunction defined by $H(x) = \{h_1(x), \dots, h_n(x)\}$, for all $x \in D$. Then, the multifunction H is an n -function. Indeed, if there exists $x_0 \in D$ and $i, j = 1, \dots, n$ with $i \neq j$, such that $h_i(x_0) = h_j(x_0)$, then,

$$\|g_i(x_0) - g_j(x_0)\| \leq \|g_i(x_0) - h_i(x_0)\| + \|h_j(x_0) - g_j(x_0)\| < 2\gamma.$$

So, we get $\gamma_{(i,j)}(G) < \gamma(G)$. This is a contradiction and H is an n -function. Now, since for all $i = 1, \dots, n$ and for every $x \in D$, we have, $\|g_i(x) - h_i(x)\| < \frac{1}{2}\varepsilon$. So, we deduce that we have $d_H(G, H) < \varepsilon$. Thus, H is ε -near to G . On the other hand, we know that $Fix(H) = \bigcup_{i=1}^{i=n} Fix(h_i)$. As for all $i = 1, \dots, n$ the set $Fix(h_i)$ is finite, hence we conclude that the multifunction H has only a finite number of fixed points.

Now, we are ready to give the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $\varepsilon > 0$ be given and $G : D \rightarrow 2^C$ be a n -valued continuous multifunction. Then from [13, p.76], we know that $\gamma(G) > 0$. Set $\delta = \min(\frac{1}{4}\varepsilon, \frac{1}{2}\gamma(G))$. So, $\delta > 0$. By Lemma 2.7, there exist a simplex S_ε in C and a continuous map $h_\varepsilon : C \rightarrow S_\varepsilon$ such that $\|h_\varepsilon(x) - x\| < \delta$ for every $x \in C$. Now, we define a new continuous multifunction $H_\varepsilon : D \rightarrow C$ by setting:

$$H_\varepsilon(x) = (h_\varepsilon \circ G)(x), \text{ for all } x \in D.$$

Let $x \in D$ such that $G(x) = \{y_1, \dots, y_n\}$, then $H_\varepsilon(x) = \{h_\varepsilon(y_1), \dots, h_\varepsilon(y_n)\}$. Assume that there is $i, j \in \{1, \dots, n\}$ such that $i \neq j$ and $h_\varepsilon(y_i) = h_\varepsilon(y_j)$. So, we get

$$\|y_i - y_j\| \leq \|y_i - h_\varepsilon(y_i)\| + \|h_\varepsilon(y_j) - y_j\|.$$

Thus, we have $\|y_i - y_j\| < 2\delta$. Hence, $\|y_i - y_j\| < \gamma(G)$. Thus is a contradiction and we deduce that the multifunction H_ε is n -valued. Now, since for all $i = 1, \dots, n$ we have $\|y_i - h_\varepsilon(y_i)\| < \frac{1}{4}\varepsilon$, then we get $d_H(G, H_\varepsilon) < \frac{1}{2}\varepsilon$. Thus, H_ε is $\frac{1}{2}\varepsilon$ -near to G . By Lemma 2.3, the multifunction $H_\varepsilon : D \rightarrow 2^C$ is an n -function. Using this and Proposition 4.2, we deduce that there exists an n -function $K_\varepsilon : D \rightarrow 2^C$ which is $\frac{1}{2}\varepsilon$ -near to H_ε and has only a finite number of fixed points. As

$$d_H(G, K_\varepsilon) \leq d_H(G, H_\varepsilon) + d_H(H_\varepsilon, K_\varepsilon).$$

So, we get $d_H(G, K_\varepsilon) < \varepsilon$. Therefore, we conclude that the multifunction $K_\varepsilon : D \rightarrow 2^C$ is ε -near to G and has only a finite number of fixed points.

As consequences of Theorem 4.1, we get the following.

Corollary 4.3 *Let C_i be a nonempty compact convex subsets of an F -space $(E, \|\cdot\|)$ for $i = 1, \dots, n$ such that $\cap_{i=1}^n C_i \neq \emptyset$. Let $k \in \{1, \dots, n\}$ and let $G : \cup_{i=1}^n C_i \rightarrow C_k$ be a continuous n -valued multifunction and $\varepsilon > 0$ be given. Then, there exists a continuous n -valued multifunction $K_\varepsilon : \cup_{i=1}^n C_i \rightarrow C_k$ which is ε -near to G and has only a finite number of fixed points.*

Corollary 4.4 *Let C be a nonempty compact convex subset of an F -space $(E, \|\cdot\|)$, $G : C \rightarrow C$ be a continuous n -valued multifunction and $\varepsilon > 0$ be given. Then,*

- (i) *Fix(G) has at least n fixed points ;*
- (ii) *there exists a continuous n -valued multifunction $K_\varepsilon : C \rightarrow C$ which is ε -near to G and has only a finite number of fixed points.*

References

- [1] Baillon, J.B. and Rallis, N.E.: Not too many fixed points, Contemporary Mathematics Vol. 72, 21-24 (1988).
- [2] Borsuk, K.: Theory of retracts, Monografie Matematyczne t. 44, Warszawa 1967.
- [3] Brown, R.F.: The Lefschetz fixed point Theorem, Scott, Foresman, and Company, Glenview, Illinois 1971.

- [4] Cauty, R.: Solution du problème de point fixe de Schauder [Solution of Schauder's fixed point problem], Fund. Math. 170, no. 3, 231-246 (2001).
- [5] Cobzas, S.: Fixed point theorems in locally convex spaces-the Schauder mapping method, Fixed Point Theory Appl. 2006, Art. ID 57950, 13 pp.
- [6] Dobrowolski, T.: Revisiting Cautys proof of the Schauder conjecture, Abstr. Appl. Anal. no. 7, 407433 (2003).
- [7] Dugundji, J.: Absolute neighborhood retracts and local connectedness in arbitrary metric spaces, Compositio Math. 13, 229-296 (1958).
- [8] Hopf, H.: Über die algebraische Anzahl von Fixpunkten, Math. Z. 29, 493-524 (1929).
- [9] Hu, S.T.: Theory of Retracts, Wayne state University Press, Detroit, Michigan 1959.
- [10] Kalton, N.J., Peck, N.T. and Roberts, J.W.: An F-space sampler, London Mathematical Society Lecture Note Series vol. 89, Cambridge University Press, Cambridge 1984.
- [11] Schrimmer, H.: Fix-finite approximation of n -valued multifunctions, Fundamenta Mathematicae CXXI, 73-80 (1984).
- [12] Stouti, A.: Fix-finite approximation property in normed vector spaces, Extracta Math. 17, no. 1, 123-130 (2002).
- [13] Stouti, A.: A fix-finite approximation theorem, Rocky Mountain J. Math. 34, no. 4, 1507-1517 (2004).

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